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## NORMAL HILBERT POLYNOMIALS

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### 1. Normal Hilbert Polynomials.

This note is a short summary of my recent work [5]. Throughout this note  $(A, m)$  will be a Cohen-Macaulay local ring of dimension  $d \geq 2$ , and  $I$  will be a parameter ideal for  $A$ , i.e.  $I$  is an  $m$ -primary ideal generated by  $d$  elements. Assume that  $A$  is analytically unramified and  $A/m$  is infinite. For an ideal  $J$  in  $A$ ,  $\bar{J}$  denotes the integral closure of  $J$ , i.e.,  $\bar{J} = \{x \in A \mid x^n + a_1 x^{n-1} + \cdots + a_n = 0 \text{ for some } a_i \in J^i\}$ .

It is well known that there exist uniquely determined integers  $\bar{e}_0(I), \dots, \bar{e}_d(I)$  such that

$$\text{length}_A(A/\bar{I}^{n+1}) = \bar{e}_0(I) \binom{n+d}{d} - \bar{e}_1(I) \binom{n+d-1}{d-1} + \cdots + (-1)^d \bar{e}_d(I)$$

for all large  $n$ . We say that

$$P(I, n) = \bar{e}_0(I) \binom{n+d}{d} - \bar{e}_1(I) \binom{n+d-1}{d-1} + \cdots + (-1)^d \bar{e}_d(I)$$

is the *normal Hilbert polynomial* of  $I$ .  $\bar{e}_0(I)$  is a well-known number called the multiplicity of  $I$  i.e.,  $\bar{e}_0(I) = e(I) = \text{length}_A(A/I)$ . Our purpose of this note is to report some properties of  $\bar{e}_1(I)$ ,  $\bar{e}_2(I)$  and  $\bar{e}_3(I)$ . Our results are contained in the following two theorems.

THEOREM 1. (1)  $\bar{e}_1(I) - \text{length}_A(\bar{I}/I) \geq \text{length}_A(\bar{I}^2/\bar{I}\bar{I})$ , and the equality holds if and only if  $\bar{I}^{n+2} = I^n \bar{I}^2$  for every  $n \geq 0$ .

(2)  $\bar{e}_2(I) \geq \bar{e}_1(I) - \text{length}_A(\bar{I}/I)$ , and the equality holds if and only if  $\bar{I}^{n+2} = I^n \bar{I}^2$  for every  $n \geq 0$ .

THEOREM 2. (1)  $\bar{e}_3(I) \geq 0$ , and if  $\bar{e}_3(I) = 0$ , then  $\bar{I}^{n+2}$  is contained in  $I^n$  for every  $n \geq 0$ .

(2) Assume that  $A$  is Gorenstein and  $\bar{I} = m$ . Then  $\bar{e}_3(I) = 0$  if and only if  $\bar{I}^{n+2} = I^n \bar{I}^2$  for every  $n \geq 0$ .

Ooishi called the number  $\bar{e}_1(I) - \text{length}_A(\bar{I}/I)$  the *normal sectional genus* of  $I$  and denoted it by  $\bar{g}_s(I)$ .

In two dimensional case, Huneke remarked in his paper that

$$\begin{aligned} \text{length}_A(A/\bar{I}^{n+1}) &= \text{length}_A(A/I) \binom{n+2}{2} \\ &\quad - \left( \sum_{r \geq 0} \text{length}_A(\bar{I}^{r+1}/I\bar{I}^r) \right) \binom{n+1}{1} + \sum_{r \geq 1} \text{length}_A(\bar{I}^{r+1}/I\bar{I}^r) r \end{aligned}$$

for all large  $n$  ([3]). Therefore

$$\begin{aligned} \sum_{r \geq 1} \text{length}_A(\bar{I}^{r+1}/I\bar{I}^r) r &\geq \bar{g}_s(I) = \sum_{r \geq 1} \text{length}_A(\bar{I}^{r+1}/I\bar{I}^r) \\ &\geq \text{length}_A(\bar{I}^2/I\bar{I}); \end{aligned}$$

thus our Theorem 1 is a natural generalization of this fact to high dimensional case, and the main difficulty is how we reduce the problem to two dimensional case.

## 2. Key Theorem.

Detailed studies in  $\bar{e}_i(I)$ 's are based on the following theorem and lemma.

**THEOREM 3.** *There exists a system of generators  $x_1, \dots, x_d$  of  $I$  such that, if we put  $C = A(T)/(\sum_i x_i T_i)$  and  $J = IC$ , where  $A(T) = A[T]_{m[T]}$  with  $T = (T_1, \dots, T_d)$   $d$  indeterminates, then*

- (1)  $\bar{J}^n \cap A = \bar{I}^n$  for every  $n \geq 0$ ;
- (2)  $\bar{J} = \bar{I}C$ ;
- (3)  $\bar{J}^n = \bar{I}^n C \cong \bar{I}^n A(T)/(\sum_i x_i T_i) \bar{I}^{n-1} A(T)$  for all large  $n$ ;
- (4)  $C$  is normal if  $A$  is analytically normal and  $\dim A \geq 3$ .

**LEMMA 4.** *Choose a system of minimal generators  $x_1, \dots, x_d$  of  $I$ , and put  $B = A[x_1/x_2]$ ,  $R = R(A, I) = A[It, t^{-1}]$  (the Rees ring of  $I$ ),  $D = R_P$ , where  $P = (t^{-1}, m)R$ , and  $nD$  = the maximal ideal of  $D$ . Let  $h : B \rightarrow D$  be a canonical homomorphism which maps  $x_1/x_2$  to  $x_1 t/x_2 t$ . Then*

- (0)  $nD \cap B = m[x_1/x_2]$ .

*We here put  $C = Bm[x_1/x_2]$  and  $J = IC$ . Then*

- (1)  $\bar{J}^n \cap A = \bar{I}^n$  for every  $n \geq 0$ ,
- (2)  $\bar{J} = \bar{I}C$  and

(3)  $\overline{I^{n+r}} \subseteq \overline{I^n}$  for every  $n \geq 0$  if and only if  $\overline{J^{n+r}} \subseteq \overline{J^n}$  for every  $n \geq 0$ .

Applying the theorem (and the lemma), we have the following results.

PROPOSITION 5. *With the same notation as in Theorem 3, we have the following assertions.*

- (1)  $\bar{e}_i(I) = \bar{e}_i(J)$  for every  $i \leq d-1$ ;
- (2)  $\bar{g}_s(I) = \bar{g}_s(J)$ ;
- (3)  $\overline{I^2 A(T)} / \overline{I \bar{I} A(T)}$  is a submodule of  $\overline{J^2 / J \bar{J}}$ , in particular  $\text{length}_A(\overline{I^2 / I \bar{I}}) \geq \text{length}_C(\overline{J^2 / J \bar{J}})$ .
- (4)  $\overline{I^{n+r}}$  is contained in  $I^n$  for every  $n \geq 0$  if and only if  $\overline{J^{n+r}}$  is contained in  $J^n$  for every  $n \geq 0$ .

PROOF. We put  $z = \sum_i x_i T_i$  for simplicity. (1) and (2) follow from Theorem 3. (3): It is enough to show that  $\overline{I^2 A(T)} \cap (\overline{I \bar{I}}, z)A(T) = \overline{I \bar{I} A(T)}$ .  $\overline{I^2 A(T)} \cap (\overline{I \bar{I}}, z)A(T) = \overline{I \bar{I} A(T)} + \overline{I^2 A(T)} \cap zA(T) = \overline{I \bar{I} A(T)} + \overline{I A(T)}z = \overline{I \bar{I} A(T)}$ . (4) follows from Lemma 4(3).

It is known that  $\bar{e}_2(I) \geq \bar{g}_s(I)$  if  $\dim A = 2$ ; therefore by the induction on  $d = \dim A$ , we have

COROLLARY 6.  $\bar{e}_2(I) \geq \bar{g}_s(I)$ .

As proved in [4, Proposition 10],

$$\begin{aligned}
 (*) \text{length}_A(A / \overline{I^{n+1}}) &\geq \text{length}_A(A / \overline{I^{n+1} \bar{I}^2}) \\
 &= \text{length}_A(A / I) \binom{n+d}{d} - (\text{length}_A(\bar{I} / I) + \text{length}_A(\overline{I^2 / I \bar{I}})) \binom{n+d-1}{d-1} \\
 &\quad + \text{length}_A(\overline{I^2 / I \bar{I}}) \binom{n+d-2}{d-2}
 \end{aligned}$$

for all  $n$ . Therefore

$$\begin{aligned}
 \bar{e}_1(I) &\geq \text{length}_A(\bar{I} / I) + \text{length}_A(\overline{I^2 / I \bar{I}}) \text{ i.e.,} \\
 \bar{g}_s(I) &= \bar{e}_1(I) - \text{length}_A(\bar{I} / I) \geq \text{length}_A(\overline{I^2 / I \bar{I}}).
 \end{aligned}$$

We give here the proof of (1) in Theorem 1.

PROPOSITION 7. (1)  $\bar{g}_s(I) \geq \text{length}_A(\overline{I^2 / I \bar{I}})$ .

(2)  $\bar{g}_s(I) = \text{length}_A(\bar{I}^2/I\bar{I})$  if and only if  $\overline{I^{n+2}} = I^n \bar{I}^2$  for every  $n \geq 0$ .

PROOF. (2) We may assume that  $A/m$  is infinite. We use the induction on  $d = \dim A$ . If  $d = 2$ , the assertion clearly holds. So assume that  $d > 2$ . If part follows from (\*). only if part: Choose a system of generators  $x_1, \dots, x_d$  of  $I$  satisfying the conditions of Theorem 3, and put  $z = \sum_i x_i T_i$ ,  $C = A(T)/zA(T)$  and  $J = IC$ . Since  $\text{length}_A(\bar{I}^2/I\bar{I}) \leq \text{length}_C(\bar{J}^2/J\bar{J}) \leq \bar{g}_s(J) = \bar{g}_s(I)$  by Proposition 5, we have  $\text{length}_C(\bar{J}^2/J\bar{J}) = \bar{g}_s(J)$  and  $\bar{J}^2 = \bar{I}^2 C$ . Thus  $\overline{J^{n+2}} = J^n \bar{J}^2$  for every  $n \geq 0$ . Then  $\overline{I^{n+2}A(T)} = \overline{I^{n+2}A(T)}$  is contained in  $I^n \bar{I}^2 A(T) + zA(T)$ , and hence  $\overline{I^{n+2}A(T)} = \overline{I^{n+2}A(T)} \cap (I^n \bar{I}^2 A(T) + zA(T)) = I^n \bar{I}^2 A(T) + z \overline{I^{n+1}A(T)}$ . By the induction on  $n$ ,  $\overline{I^{n+2}A(T)} = I^n \bar{I}^2 A(T)$ , and therefore  $\overline{I^{n+2}} = I^n \bar{I}^2$ .

As we remarked in [4, Proposition 3],

(\*\*) if  $\overline{I^{n+2}} = I^n \bar{I}^2$  for every  $n \geq 0$ , then  $G = R'/t^{-1}R'$  is Cohen-Macaulay.

Since

$$\begin{aligned} \text{length}_A(A/\overline{I^{n+1}}) &= \text{length}_A(A/I^{n+1}) \\ &\quad - \sum_{0 \leq r \leq n} \text{length}_A(I^{n-r} \overline{I^{r+1}}/I^{n-r+1} \bar{I}^r) \end{aligned}$$

and

$$\begin{aligned} &\text{length}_A(I^{n-r} \overline{I^{r+1}}/I^{n-r+1} \bar{I}^r) \\ &\leq \text{length}_A((\overline{I^{r+1}}/I\bar{I}^r) \otimes (I^{n-r}/I^{n-r+1})) \\ &= \text{length}_A(\overline{I^{r+1}}/I\bar{I}^r) \binom{n-r+d-1}{d-1} \\ &= \text{length}_A(\overline{I^{r+1}}/I\bar{I}^r) \binom{n+d-1}{d-1} - r \binom{n-r+d-2}{d-2} + \text{lower degree terms}, \end{aligned}$$

we have

$$\begin{aligned} \bar{e}_1(I) &\leq \sum_{r \geq 0} \text{length}_A(\overline{I^{r+1}}/I\bar{I}^r) \text{ i.e.,} \\ \bar{g}_s(I) &= \bar{e}_1(I) - \text{length}_A(\bar{I}/I) \leq \sum_{r \geq 1} \text{length}_A(\overline{I^{r+1}}/I\bar{I}^r). \end{aligned}$$

PROPOSITION 8. (1)  $\bar{g}_s(I) \leq \sum_{r \geq 1} \text{length}_A(\overline{I^{r+1}}/I\bar{I}^r)$ .

(2) If  $\text{depth}_M R' \geq d$ , then  $\bar{g}_s(I) = \sum_{r \geq 1} \text{length}_A(\overline{I^{r+1}}/I\bar{I}^r)$  and  $\bar{e}_2(I) = \sum_{r \geq 1} \text{length}_A(\overline{I^{r+1}}/I\bar{I}^r)r$ , where  $M = (t^{-1}, It)R$ .

### 3. $\bar{e}_2(I)$ .

In this section, we shall give the proof of (2) in Theorem 1. By Corollary 6, the assertion remained to be proved is the following

PROPOSITION 9.  $\bar{e}_2(I) = \bar{g}_s(I)$  if and only if  $\overline{I^{n+2}} = I^n \bar{I}^2$  for every  $n \geq 0$ .

If part of the above proposition clearly holds by (\*). Only if part follows from the following proposition.

PROPOSITION 10. Assume that  $d \geq 3$ , and choose a system of generators  $x_1, \dots, x_d$  satisfying the conditions of Theorem 1, and put  $z = \sum_i x_i T_i$ ,  $C = A(T)/zA(T)$  and  $J = IC$ . If  $\overline{J^{n+2}} = J^n \bar{J}^2$  for every  $n \geq 0$ , then  $\overline{I^{n+2}} = I^n \bar{I}^2$  for every  $n \geq 0$ .

PROPOSITION 11. Let  $r$  be either 1 or 2. Then the following assertions are equivalent.

- (1)  $\overline{I^{n+r}} = I^n \bar{I}^r$  for every  $n \geq 0$ .
- (2)  $[H_N^i(R')]_j = 0$  for  $i + j \geq r + 1$ .
- (3)  $[H_N^i(R')]_j = 0$  for  $i + j = r + 1$ .

*Proof of Proposition 10:* Assume that  $d \geq 3$ , and choose a system of generators  $x_1, \dots, x_d$  satisfying the conditions of Theorem 3, and put  $z = \sum_i x_i T_i$ ,  $C = A(T)/zA(T)$ ,  $J = IC$ , as in Theorem 3.  $S = A(T) \otimes R (= R(A(T), IA(T)))$ ,  $S' = A(T) \otimes R' (= R'(A(T), IA(T)))$ ,  $N' = ItS$ ,  $M' = (t^{-1}, It)S'$ ,  $F = S/zS (= R(C, J))$  and  $F' = \sum \bar{I}^n C t^n (= R'(C, J))$ . Suppose that  $[H_{N'}^i(F')]_j = 0$  for  $i + j = 3$ . We shall prove  $[H_N^i(R')]_j = 0$  for  $i + j = 3$ . Since  $A \rightarrow A(T)$  is faithfully flat,  $H_N^i(R') = H_{N'}^i(S')$ ; thus it is sufficient to prove that  $[H_{N'}^i(S')]_j = 0$  for  $i + j = 3$ . We first prove that  $[H_{N'}^{i-1}(S'/ztS')]_{j+1} = 0$  for  $3 \leq i \leq d$  and  $j \geq 3 - i$ . If this is proved, then  $([H_N^i(S')]_j = [H_{N'}^i(S'(-1))]_{j+1} \xrightarrow{zt} [H_{N'}^i(S')]_{j+1})$  is injective; since every element of  $H_{N'}^i(S')$  is annihilated by some power of  $zt$ ,  $[H_N^i(S')]_j$  must be 0. Since  $\dim F'/(S'/ztS') = 0$ , we have  $H_{N'}^1(S'/ztS') = F'/(S'/ztS')$  and  $H_{N'}^i(S'/ztS') = H_{N'}^i(F')$  for  $i \geq 2$ . Therefore, for  $3 \leq i \leq d$ ,  $H_{N'}^{i-1}(S'/ztS')_{j+1} = H_{N'}^{i-1}(F')_{j+1} = 0$ , since  $F'$  is a Cohen-Macaulay ring. We next prove that  $H_{N'}^2(S')_1 = 0$ . It is known that  $H_{N'}^2(S')_0 = 0$ ; therefore

$0 = H_{N'}^2(S')_0 \xrightarrow{zt} H_{N'}^2(S')_1 \longrightarrow H_{N'}^2(S'/ztS')_1 = 0$ ; hence  $H_{N'}^2(S')_1 = 0$ . It is also known that  $H_{M'}^0(S') = H_{M'}^1(S') = 0$  and  $H_{M'}^i(S') = H_{N'}^i(S')_i = 0, 1$ . Therefore  $H_{N'}^i(S') = 0$  for  $i = 0, 1$ .

#### 4. $\bar{e}_3(I)$ .

In general, it is known that

$$(-1)^d \bar{e}_d(I) = F(0) = \sum_i (-1)^i \text{length}_A([H_N^i(R')]_0).$$

If  $d = 3$ , then  $\bar{e}_3(I) = \text{length}_A([H_N^3(R')]_0) = \text{length}_A(H^2(X, O_X))$ , because  $[H_N^2(R')]_0 = [H_M^2(R')]_0 = 0$ . Therefore by Proposition 5,

$$(***) \bar{e}_3(I) \geq 0.$$

It is then natural to ask when  $\bar{e}_3 = 0$ . It follows from [4, Appendix 2] that there exists a canonical graded homomorphism  $\alpha : H_N^d(R') \longrightarrow H_m^d(A)[t, t^{-1}]$ . We denote by  $\alpha_j$  the graded part of degree  $j$  of  $\alpha$ . Then we have

LEMMA 12.  $\alpha_j = 0$  (i.e.,  $[H_M^d(R')]_j = [H_N^d(R')]_j$ ) if and only if  $\overline{I^{n+d-1+j}} \subseteq I^n$  for every  $n \geq 0$ .

*Proof of Theorem 2.* (1): By Proposition 5, we may assume that  $d = \dim A = 3$ . Then the assertion follows from (\*\*\*) and Lemma 12. (2): This follows from the following proposition.

PROPOSITION 13. Assume that  $A$  is a Gorenstein local ring and  $\bar{I}^2 = I\bar{I}$ . If  $\overline{I^{n+2}}$  and  $m\bar{I}^{n+1}$  are contained in  $I^n$  for every  $n \geq 0$ , then  $\text{length}_A(\bar{I}^2/I\bar{I}) = 1$  and  $\overline{I^{n+2}} = I^n \bar{I}^2$  for every  $n \geq 0$ .

PROOF. Since  $m\bar{I}^{n+1} \supseteq I^n$ , we have  $(I^n : m)/I^n \supseteq (\overline{I^{n+1}} + I^n)/I^n = \overline{I^{n+1}}/I^n \bar{I}$ , and hence  $\text{length}_A(\overline{I^{n+1}}/I^n \bar{I}) \leq \text{length}_A((I^n : m)/I^n) = \binom{n-1+d-1}{d-1}$ , because  $A$  is Gorenstein. Therefore

$$\begin{aligned} & \text{length}_A(A/\overline{I^{n+1}}) \\ &= \text{length}_A(A/I^{n+1}) - \text{length}_A((I^n \bar{I}/I^{n+1}) - \text{length}_A(\overline{I^{n+1}}/I^n \bar{I}) \\ &\geq \text{length}_A(A/I) \binom{n+d}{d} - \text{length}_A(\bar{I}/I) \binom{n+d-1}{d-1} - \binom{n-1+d-1}{d-1} \\ &= \text{length}_A(A/I) \binom{n+d}{d} - (\text{length}_A(\bar{I}/I) + 1) \binom{n+d-1}{d-1} + \binom{n+d-2}{d-2}. \end{aligned}$$

(We have already proved in [4, Proposition 10] that  $\text{length}_A((I^n \bar{I}/I^{n+1}) = \text{length}_A(\bar{I}/I) \binom{n+d-1}{d-1}$ .) Thus by (\*),  $\text{length}_A(\bar{I}^2/I\bar{I}) = 1$  and  $\text{length}_A(A/\overline{I^{n+1}}) = \text{length}_A(A/I) \binom{n+d}{d} - (\text{length}_A(\bar{I}/I) + 1) \binom{n+d-1}{d-1} + \binom{n+d-2}{d-2}$ , and in particular,  $\overline{I^{n+1}} = I^{n-1} \bar{I}^2$ .

It is natural to ask whether the assertion (2) in Theorem 2 is true for any parameter ideals.

CONJECTURE: Assume that  $A$  is Gorenstein and  $d = \dim A \geq 3$ . Then  $\bar{e}_3(I) = 0$  if and only if  $\overline{I^{n+2}} = I^n \bar{I}^2$  for every  $n \geq 0$ .

Assume that  $d = 3$  and  $\bar{e}_3(I) = 0$ : By Proposition 11, if  $[H_M^2(R')]_1 (= [H_N^2(R')]_1) = 0$ , then  $\overline{I^{n+2}} = I^n \bar{I}^2$  for every  $n \geq 0$ .

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